U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

SOME RESULTS IN NONLINEAR PROGRAMMING
PART II

R. M. Thrall

RM-935

5 September 1952

Assigned to ______________________

This is a working paper. It may be expanded, modified, or withdrawn at any time.

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

Report Documentation Page			*Form Approved* *OMB No. 0704-0188*

1. REPORT DATE **05 SEP 1952**	2. REPORT TYPE	3. DATES COVERED **00-00-1952 to 00-00-1952**
4. TITLE AND SUBTITLE **Some Results in Nonlinear Programming Part II**		5a. CONTRACT NUMBER 5b. GRANT NUMBER 5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)		5d. PROJECT NUMBER 5e. TASK NUMBER 5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Rand Corporation,Project Air Force,1776 Main Street, PO Box 2138,Santa Monica,CA,90407-2138**		8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR'S ACRONYM(S) 11. SPONSOR/MONITOR'S REPORT NUMBER(S)
12. DISTRIBUTION/AVAILABILITY STATEMENT **Approved for public release; distribution unlimited**		
13. SUPPLEMENTARY NOTES		
14. ABSTRACT		
15. SUBJECT TERMS		

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**16**	

Summary: The solution of a minimization problem is obtained in the vector case; its properties are studied and applied to a maximization problem.

SOME RESULTS IN NONLINEAR PROGRAMMING, PART II

R. M. Thrall

§1. Introduction.

In the present note we consider a class of maximization and minimization problems which fall under the general heading of nonlinear programming. The functions which enter are assumed to satisfy enough differentiability conditions so that methods of calculus can be applied and so that existence of at least one solution is trivial. This note is an extension and generalization of RM-909. J. Danskin has considered a similar problem for the functional case in RM-618 and this has been generalized by J. Danskin and H. Kahn in a forthcoming memorandum.

§2. Notation and Geometric Preliminaries.

Let V_n be the space of all real vectors $v = (v_1, \ldots, v_n)$. For any vector v we define v_0 to be the sum of the components of v. A vector v is said to be positive, written $v > 0$ ($0 = (0, \ldots, 0)$) if each component is positive. We write

$v \geqq 0$ if each component is non negative and write $v \geq 0$ if $v \geqq 0$ but $v \neq 0$. If $v_0 = 1$ and $v \geq 0$ we call v a probability vector.

For any $x_0 > 0$, we denote by $\Gamma = \Gamma(x_0)$ the set of all vectors x in V_n for which $x \geq 0$ and $x_1 + \ldots + x_n = x_0$. Let E be any subset of $\{1, \ldots, n\}$. We denote by Γ_E the set of all vectors x in Γ for which $x_i \neq 0$, i in E and $x_i = 0$, i not in E. We call the subsets Γ_E the <u>faces</u> of Γ. Clearly, Γ is partitioned by its faces.

§3. <u>Functions of Type A</u>.

Let $q(t)$ be a real valued function defined for $t \geqq 0$ and with the following properties:

$$(A) \quad \begin{cases} \text{(i)} & q(t) \text{ monotone decreasing} \\ \text{(ii)} & q''(t) > 0 \text{ for all } t > 0 \\ \text{(iii)} & \lim_{t \to \infty} q(t) = 0 \\ \text{(iv)} & q(0) = 1 \end{cases}$$

Actually, properties (iii) and (iv) could be replaced by the single assumption $\lim_{t \to \infty} q(t) > -\infty$, but then by a normalization we could regain (iii) and (iv). A function which satisfies conditions A is said to be of type A. We observe that one consequence of (i) and (ii) is that $q'(t)$ is a monotone increasing function with negative values.

§ 4. The Minimization Problem.

Let P be a positive probability vector, let $q_1(t), \ldots, q_n(t)$ be functions of type A and consider minimization of the function

$$(1) \qquad f(x,p) = \sum_{i=1}^{n} p_i q_i(x_i)$$

for vectors x in $\Gamma(x_0)$. Since $f(x,p)$ is continuous in x and since $\Gamma(x_0)$ is compact the minimum exists; we denote it by

$$(2) \qquad g(x_0) = \min_{x \in \Gamma(x_0)} f(x,p) \ .$$

Let $\bar{x}$ be a minimizing vector and suppose that $\bar{x} \in \Gamma_E$. Let $j_0 \in E$ and write

$$(3) \qquad x_{j_0} = 1 - \sum_{j \neq j_0} x_j \ .$$

For $j \neq j_0$, we have

$$(4) \qquad \frac{\partial f(x,p)}{\partial x_j} = p_j q_j'(x_j) - p_{j_0} q_{j_0}'(x_{j_0}).$$

In particular, for $x = \bar{x}$ we must have

$$(5) \qquad \left.\frac{\partial f}{\partial x_j}\right]_{x=\bar{x}} \geqq 0 \qquad j \neq j_0$$

with equality holding for $j \in E$.

From (4) this gives

$$p_j q_j'(\bar{x}_j) = p_{j_o} q_{j_o}'(\bar{x}_{j_o}) \equiv \mu_E \qquad (j \in E) \tag{6}$$

and

$$p_j q_j'(0) \geqq \mu_E \qquad (j \notin E). \tag{7}$$

Let

$$\rho_j = p_j q_j'(0) \qquad (j=1,\ldots,n) . \tag{8}$$

Now since $q_j'(t)$ is a monotone increasing function, it follows from (6), (7), and (8) that

$$\begin{cases} \mu_E > \rho_j & (j \in E) \\ \mu_E \leqq \rho_j & (j \notin E). \end{cases} \tag{9}$$

Next, we arrange the indices so that

$$\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_n, \tag{10}$$

and define $\rho_0 = 0$, $\rho_{n+1} = -\infty$.

It then follows from (9) that E must be one of the sets

$$E_h = \{h,\ldots,n\} \quad (h=1,\ldots,n) . \tag{11}$$

This has already cut down the possible locations Γ_E for minimizing vectors from 2^n to n. We next show that there is only one h for which E_h satisfies (9). We write μ_h for

μ_{E_h} and Γ_h for Γ_{E_h}. Then (9) is replaced by

$$(12) \qquad \rho_{h-1} \geqq \mu_h > \rho_h$$

where $1 \leqq h \leqq n + 1$.

Let $r_j(w)$ be the inverse of $q_j'(t)$, i.e.,

$$(13) \qquad r_j(q_j'(t)) = t, \quad q_j'(r_j(w)) = w \qquad (j=1,\ldots,n).$$

The domain of r_j is $\rho_j/p_j \leqq w < 0$ and its range is $0 \leqq r_j(w) < \infty$.

If x is a maximizing vector which is in Γ_h we have as a consequence of (6) that

$$(14) \qquad \begin{cases} \bar{x}_j = 0 & (j < h) \\ \bar{x}_j = r_j(\mu_h/p_j) & (j \geqq h) \end{cases}$$

where $\mu_h = \mu_h(x_0)$ is defined as a function of x_0 by the equation

$$(15) \qquad x_0 = \sum_{j=h}^{n} r_j(\mu_h/p_j).$$

Since each $q_j(t)$ is of type A, the inverse functions $r_j(w)$ are monotone increasing. Hence equation (15) has a unique solution μ_h which is a monotone increasing function of x_0 with domain

$$x_0 \geqq y_h = \sum_{j=h}^{n} r_j(\rho_h/p_j)$$

and range $\rho_h \leqq \mu_h < 0$. Moreover, we have for $x_0 \geqq y_h$ that

$$(16) \qquad \mu_h(x_0) \leqq \mu_{h+1}(x_0)$$

and equality holds in (16) if and only if $r_j(\mu_j(x_0)/p_j) = 0$; i.e., if and only if $\mu_h(x_0) = \rho_h$. But now it follows from (15) that

$$(17) \qquad \begin{cases} \mu_h(y_h) = \mu_{h+1}(y_h) = \rho_h & (h=1,\ldots,n-1) \\ \mu_n(y_n) = \rho_n. \end{cases}$$

We set $y_0 = \infty$ and then have $0 = y_n \leqq y_{n-1} \cdots \leqq y_1 \leqq y_0$.

Formula (17) provides the clue for the determination of an h which satisfies (12). Indeed, (12) holds if and only if h satisfies the condition

$$(18) \qquad y_h < x_0 \leqq y_{h-1}.$$

Since (18) has only one solution we have established uniqueness for the minimizing vector. We summarize these results in the following theorem.

<u>Theorem 1</u>. <u>Let</u> $q_1(t),\ldots,q_n(t)$ <u>be functions of type</u> A, <u>let</u> p <u>be a positive probability vector, and let</u> x_0 <u>be a positive real number. Then the function</u>

$$f(x,p) = \sum_{i=1}^{n} p_i q_i(x_i)$$

<u>where</u> x <u>has domain</u> $\Gamma(x_0)$ <u>has a unique minimizing vector</u> $\bar{x}$

given by (14) where h is determined by (18) and μ_h by (15); the minimum value is

$$(19) \qquad f(\bar{x},p) = \sum_{j=1}^{h-1} p_j + \sum_{j=h}^{n} p_j q_j(r_j(\mu_h/p_j)).$$

The minimum value of f(x,p) can be considered as a function of x_0, viz

$$(20) \qquad g(x_0) = f(\bar{x},p).$$

For this function we have the following theorem:

Theorem 2. The minimum $g(x_0)$ of f(x,p) is a differentiable monotone decreasing function of x_0 in the interval $x_0 > 0$; moreover, $g'(x_0)$ is continuous for all $x_0 > 0$ and

$$(21) \qquad \lim_{x_0 \to 0^+} g'(x_0) = \rho_n.$$

The second derivative $g''(x_0)$ is a continuous, positive function of x_0 except possibly for the values $y_1,\ldots,y_{n-1}$; $g''(x_0)$ is continuous at y_h if and only if $\lim_{t \to 0^+} q_j''(t) = \infty$, but in any case the one-sided limits exist and are positive. Moreover, $\lim_{x_0 \to 0^+} g''(x_0)$ exists if and only if $\lim_{t \to 0} q_n''(t)$ exists.

We first establish the continuity of $g(x_0)$. We have

$$\lim_{x_0 \to y_h^+} g(x_0) = \sum_{j=1}^{h-1} p_j + \sum_{j=h}^{n} p_j q_j(r_j(\rho_t/p_j))$$

and

$$\lim_{x_0 \to y_h^-} g(x_0) = \sum_{j=1}^{h} p_j + \sum_{j=h+1}^{n} p_j q_j(r_j(\rho_t/p_j)).$$

The difference is

$$p_h - p_h q_h(r_h(\rho_h/p_h))$$

and this is zero since $r_h(\rho_h/p_h) = 0$ and $q_h(0) = 1$.

Next, for $y_h < x_0 \leqq y_{h-1}$ we have

$$(22) \qquad g'(x_0) = \sum_{j=h}^{n} p_j q_j'(r_j(\mu_h/p_j)) \cdot r_j'(\mu_h/p_j) \cdot \frac{1}{p_j} \cdot \frac{d\mu_h}{dx_0}$$

$$= \sum_{j=h}^{n} p_j \frac{\mu_h}{p_j} \cdot \frac{1}{p_j} r_j'(\mu_h/p_j) \frac{d\mu_h}{dx_0} .$$

Differentiating (15) we get

$$(23) \qquad 1 = \sum_{j=h}^{n} \frac{1}{p_j} r_j'(\mu_h/p_j) \cdot \frac{d\mu_h}{dx_0}.$$

Now from (22) and (23) we get

$$(24) \qquad g'(x_0) = \mu_h \qquad (y_h < x_0 \leqq y_{h-1}).$$

Now $\lim_{x_0 \to y_t^+} \mu_h = \lim_{x_0 \to y_t^-} \mu_h = \rho_h$ and $\lim_{x_0 \to y_n^+} \mu_h = \rho_n$.

This establishes all of the statements about $g'(x_0)$.

Finally, for $y_h < x_0 \leqq y_{h-1}$ we have

$$(25) \qquad g''(x_0) = \frac{d\mu_h}{dx_0} = 1 \Big/ \sum_{j=h}^{n} \frac{1}{p_j} r_j'(\mu_h/p_j).$$

Now, differentiating (13) we get

$$(26) \qquad r_j'(q_j'(t)\; q_j''(t) = 1$$

and then since $\mu_h = p_j q_j'(\bar{x}_j)$ $(j=t,\ldots,n)$ and since $q_j(t)$ is of type A, we have

$$(27) \qquad r_j'(\mu_h/p_j) = 1/q_j''(\bar{x}_j) > 0.$$

Now it follows readily from (25) and (27) that $g''(x_0)$ is continuous and positive except possibly at $y_1,\ldots,y_{n-1}$. Next we observe that the only term in the denominator of $\lim_{x_0 \to y_h^+} g''(x_0)$ which is not also in the denominator of $\lim_{x_0 \to y_h^-} g''(x_0)$ is

$$(28) \qquad \lim_{x_0 \to y_h^+} 1/\, p_h q_h''(\bar{x}_h) = \lim_{t \to 0} 1/p_h q_h''(t).$$

This equation establishes the statements about $g''(x_0)$ at $y_1,\ldots,y_{n-1}$. Finally, if $0 < x_0 \leqq y_{n-1}$ we have

$$(29) \qquad g''(x_0) = p_n q_n''(x_0)$$

and the final statement follows from this formula. We have also established the following corollary:

Corollary 1. *If $q_1(t),\ldots,q_n(t)$ are functions of type A and if $\lim_{t\to 0^+} q_i''(t) = \infty$ $(i=1,\ldots,n)$ then $g(x_0)$ is also of type A with $\lim_{x_0\to 0^+} g''(x_0) = \infty$.*

§5. Functions of Type B.

If conditions A are replaced by the weaker conditions

$$
(B)\quad \begin{cases} \text{(i), (iii), (iv) same as for A} \\ \text{(ii): } q''(x_0) > 0 \text{ for all } t > 0 \text{ except for a finite number of points and at these points the one sided limits exist and are positive; and } q'(x_0) \text{ is continuous for all } t > 0. \end{cases}
$$

we have a new class of functions which we call functions of type B. Theorem 2 states essentially that $g(x_0)$ is a function of type B. Actually, if the initial $q_i(t)$ are all of type B the conclusions of Theorems 1 and 2 still hold except that $g''(x_0)$ will have as points of discontinuity not only the y_j but also those x_0 for which there exists a j such that $\overline{x}_j$ is a point of discontinuity of $q_j''(t)$.

§6. The Inverse Power Case.

There are various ways in which the functions $q_i(t)$ of type A may be encountered. If $p(t)$ is a function which satisfies A(i), (ii), (iii) and if s is any positive vector, then the functions

$$(30) \qquad q_j(t) = \frac{p(x_j+s_j)}{p(s_j)} \qquad (j=1,\ldots,n)$$

are of type A.

In general one cannot expect explicit solutions for the y_j and $\mu_j(x_0)$. However in the special case

$$(31) \qquad p(t) = t^{-\gamma} \qquad (\gamma > 0)$$

of (30) it is possible to obtain explicit solutions. In this case we have

$$(32) \qquad \begin{cases} q_j(x_j) = s_j^{\gamma}/(s_j+x_j)^{\gamma} \\ q_j'(x_j) = -\gamma s_j^{\gamma}/(s_j+x_j)^{\gamma+1} \end{cases} \qquad (j=1,\ldots,n)$$

and hence

$$(33) \qquad \begin{cases} \rho_j = -\gamma p_j/s_j & (j=1,\ldots,n) \\ r_j(w) = \sqrt[\gamma+1]{-\gamma s_j{}^{\gamma}/w} \; -s_j \end{cases}$$

Then solving (15) for μ_h we get

$$(34) \qquad \mu_h = \mu_h(x_0) = -(\tau_h/(x_0+\sigma_h))^{\gamma+1} \quad (h=1,\ldots,n)$$

where

$$(35) \qquad \begin{cases} \tau_h = \sum_{j=h}^{n} \sqrt[\gamma+1]{\gamma s_j{}^{\gamma} \, p_j} \\ \sigma_h = \sum_{j=h}^{n} s_j \, . \end{cases}$$

Next, from (34) with $\mu_h = \rho_h$, $x_0 = y_h$ we get

$$(36) \qquad y_h = -\sigma_h + \tau_h / \sqrt[\gamma+1]{-\rho_h} \qquad (h=1,\ldots,n).$$

If h is now determined according to (18) we get for the maximizing vector $\bar{x}$

$$(37) \qquad \begin{cases} \bar{x}_j = 0 & (j < h) \\ \bar{x}_j = \dfrac{(x_0 + \sigma_h)^{\gamma+1}}{\tau_h} \sqrt{\gamma s_j^{\gamma} p_j} - s_j & (j \geqq h) \end{cases}$$

and

$$(38) \qquad g(x_0) = \sum_{j=1}^{h-1} p_j + \frac{\tau_h^{\gamma+1}}{\gamma (x_0 + \sigma_h)^{\gamma}} .$$

§7. A Maximization Problem.

Let x,p and f(x,p) be as in §4. We consider a function

$$(39) \qquad F(x,p) = K(x_0, f(x,p))$$

where for each x_0 $K(x_0,t)$ is monotone decreasing in t. Thus to maximize F for x in $\Gamma(x_0)$ we merely choose the $\bar{x}$ which minimizes f. However, if the maximization is to be for all x with $x \geq 0$ or for all x with $x_0 \leqq w$, one can proceed as follows. For each x_0 let $G(x_0)$ be the maximum of F for

$x \in \Gamma(x_0)$; i.e.,

(40) $$G(x_0) = K(x_0, g(x_0)).$$

Then the maximization of $F(x,p)$ is reduced to the scalar problem of maximizing $G(x_0)$. If $K(x_0,t)$ satisfies certain differentiability conditions we can make use of the results of Theorem 2 and apply the methods of elementary calculus to achieve the maximization. Of course, the nature of the function $g(x_0)$ will require that each interval $y_h < x_0 \leqq y_{h-1}$, be studied separately.

One choice of $K(x_0,t)$ has been treated in RM-909 and also by R. Isaacs. This is the case of (31) with $\gamma = 1$ and

(41) $$K(x_0,t) = Q(x_0+s_0)(1-t) - x_0,$$

where Q is a constant in the interval $0 < Q \leqq 1$. Interpretations for this case were given in RM-909.

A more general case is

(42) $$K(x_0,t) = H(x_0)(1-t) - x_0$$

where $H(z)$ satisfies the following conditions:

(43) $$\begin{cases} \text{(i)} & H(z) \geqq 0 \qquad\qquad 0 \leqq z < \infty \\ \text{(ii)} & H(z) - z \text{ is bounded from above for } z \geqq 0 \\ \text{(iii)} & H(z) \text{ and } H'(z) \text{ are continuous for } z \geqq 0. \end{cases}$$

These conditions are sufficient to guarantee the existence of a maximum of $G(x_0) = K(x_0, g(x_0))$. Since $g(0) = 1$, $G(0) = 0$. A sufficient condition for a positive maximum is then $G'(0) > 0$, and this maximum will occur at a point $\bar{x}_0$ for which $G'(\bar{x}_0) = 0$. If K is given by (42), we have

$$(44) \quad \begin{cases} G(x_0) = H(x_0)(1-g(x_0)) - x_0 \\ G'(x_0) = H'(x_0)(1-g(x_0) - H(x_0)g'(x_0) - 1 \\ G''(x_0) = H''(x_0)(1-g(x_0)) - 2H'(x_0)g'(x_0) - H(x_0)g''(x_0), \end{cases}$$

where the formula for $G''(x_0)$ holds only when $H''(x_0)$ and $g''(x_0)$ exist.

The case

$$(45) \qquad H(t) = c = \text{constant}$$

is easily handled. Then $G'(0) = -c\,\rho_n - 1$. If $c \leqq -1/\rho_n$ then $x_0 = 0$ is the only maximum. If $c > -1/\rho_n$, we determine h so that

$$(46) \qquad \rho_h < -\frac{1}{c} \leqq \rho_{h-1}$$

and then select $\bar{x}_0$ so that $\mu_h(\bar{x}_0) = -\frac{1}{c}$. This will be the maximizing value. This case is the only one where I have been able to find an explicit solution for arbitrary functions of the case $q_j(t)$.

(47) $$H(x_0) = Q\cdot(x_0+s_0),\quad q_j(t) = s_j^{\gamma} / (s_j+x_j)^{\gamma}$$

has been mentioned above for $\gamma = 1$. We now consider the case for $\gamma > 1$.

We have

(48) $$G''(x_0) = \frac{Q\tau_h^{\gamma+1}}{(x_0+\sigma_h)^{\gamma+1}} \left[2 - \frac{(\gamma+1)(x_0+s_0)}{(x_0+\sigma_h)} \right].$$

Since $\gamma > 1$ and $s_0 \geqq \sigma_h$ we have $G''(x_0) < 0$ for $x_0 > 0$. Hence there is a unique maximum. A simple calculation shows that

(49) $$G'(0) = \gamma Q s_0 p_n / s_n - 1$$

and hence that the maximum value of G is positive if and only if

(50) $$Q > s_n/(\gamma\, s_0 p_n).$$

In general for $y_h \leqq x \leqq y_{h-1}$ we find that

(51) $$G'(x) = -1 + Q\left\{\pi_h + (1-\frac{1}{\gamma})\,\tau_h(-\mu_h)^{\gamma/\gamma+1} + (s_0-\sigma_h)(-\mu_h)\right\}.$$

If $G'(y_h) \geqq 0$ but $G'(y_j) < 0$ for $j < h$, then the maximizing value is in the interval $y_h \leqq x < y_{h-1}$ and is given by

(52) $$x_0 = -\sigma_h + \tau_h/t$$

where t is the positive root of

(53) $$t^{\gamma+1} + \frac{\tau_h}{s_0-\sigma_h}\,(1-\frac{1}{\gamma})\,t^{\gamma} - \frac{1-Q\pi_h}{Q(s_0-\sigma_h)} = 0.$$

15/ STI-ATI-210 473 UNCLASSIFIED

18/7 Rand Corp., Santa Monica, Calif.

DECISION PROCESSES AND FUNCTIONAL EQUATIONS, by S. Karlin and Harold N. Shapiro. 3 Sep 52, 10p. (Rept. no. RM-933) (Contract AF 33(038)6413)

SUBJECT HEADINGS

Programming

Strategy

Topology

DIV: Mathematics (15)

SECT: All

DIV: Military Sciences & Operations (18)

SECT: Strategy & Tactics (7)

DIST: Copies obtainable from ASTIA-DSC

Proj. Rand

UNCLASSIFIED